

The Impact of Negative Interest Rates on Optimal Capital Injections

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Abstract

In the present paper, we investigate the optimal capital injection behaviour of an insurance company if the interest rate is allowed to become negative. The surplus process of the considered insurance entity is assumed to follow a Brownian motion with drift. The changes in the interest rate are described via a Markov-switching process. It turns out that in times with a positive rate, it is optimal to inject capital only if the company becomes insolvent. However, if the rate is negative it might be optimal to hold a strictly positive reserve. We establish an algorithm for finding the value function and the optimal strategy, which is proved to be of barrier type. Using the iteration argument, we show that the value function solves the Hamilton–Jacobi–Bellman equation, corresponding to the problem.

Key words: negative interest rate, capital injections, Markov-switching, optimal stochastic control, Hamilton–Jacobi–Bellman equation.

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1 Introduction

On the 16th of March 2016 the European Central Bank (ECB) set the key interest rate on 0%. The deposit facility rate (currently -0.4%) remains negative since the 11th of June 2014, confer [20]. It means, that instead of getting paid for depositing money into the central bank, one has to pay the central bank for it. Also, the yields on government bonds are currently close to their historical minimum. For instance, the yield on the 10-year German government bond, considered one of the safest assets in the world, sank below zero in June 2016 for the first time ever. But why would anyone buy a government bond, lacking annual payments and

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bringing back less than the amount invested? One reason is the deficit of alternative safe opportunities. Of course, a large corporation can hire guards in order to protect its cash. But doubtless, using bank services is safer and cheaper even in times of negative interest rates.

Since, insurance companies run massive portfolios of bonds, the changes in the interest rates could be crucial for their balance sheets. Intuitively, it is clear that ultra-low interest rates immensely affect the life insurance sector: the long-term promises to policyholders, made decades ago, imply a much higher interest rate and cause mismatches between assets and liabilities.

But do negative interest rates affect the value of a non-life insurance company? Typically, one assumes that there is little impact because most policies are short-termed, implying that the assets and liabilities can be properly matched. However, this perspective neglects the value of future business potential, for instance future premia (competitive markets), dividends (profitability) or capital injections (Solvency II capital requirements).

Indeed, non-life insurance premia should be based on the premise of appropriate pricing and give a “forecast” on profitability and possible dividend payments. Therefore, the premia are highly dependent on the economic markers. Also, Solvency II emphasizes the importance of incorporating all the risks, including the inflation risk and the interest rate risk, for the calculation of the capital requirement.

The crisis of 2008 and the bad situation in 2015, which is considered as the worst year since the crisis of 2008, let the economists speak of business-cycle dynamics characterized by more than one interest rate, confer for instance [19]. Mathematically one can translate the cycle dynamics into a Markov-switching model, where the interest rate switches at random times and is kept constant inbetween. This model has been widely investigated in the mathematical finance literature, confer for instance Boyarchenko and Levendorskii [5], Jiang and Pistorius [9], Duan et al. [7], Ly Vath et al. [11] or Akyildirim et al. [1]. In actuarial mathematics, some recent results on the risk theory in a Markovian environment can be found for instance in Asmussen [2] or Bäuerle [3] some optimisation problems have been investigated for example in Zhu and Yang [18] or Jiang and Pistorius [10].

Throughout the life cycle of a business, a company can face considerable economic challenges and multiple instances of financial distress. As a consequence, it might require capital injections to remain afloat. In actuarial mathematics, the term capital injections and the corresponding risk measure have been proposed in the discussion in Pafumi [13]. Further discussions can be found in Dickson and Waters [6], Eisenberg and Schmidli [8] or in Nie et al. [12]. In their study Nie et al. even assume that the capital injections do not eliminate the possibility of ruin for the insurer.

In the present paper, we assume that the considered insurance entity models its surplus via a Brownian motion with drift. The interest rate can attain a negative and a positive value, mimicking a business-cycle with two states. The target is to

minimise the value of expected discounted capital injections, under the constraint that the company is not allowed to become insolvent. It is intuitively clear that in the time periods with positive interest rates, it is optimal to inject capital just if the surplus becomes negative and just as much as is necessary to land at zero. However, in times with negative yields it might be optimal to hold the surplus above a strictly positive barrier. The heuristic explanation is that early injections appear cheaper than later payments.

The paper is organised as follows. In Section 2, we formulate the problem and investigate its well-posedness. In Section 3, we briefly consider the strategy with minimal-amount injections, identify the optimal strategy as a barrier strategy and introduce an algorithm for approximation of the value function. By barrier strategy we mean that capital injections are made to prevent the surplus drop below a certain level (barrier), depending on the actual interest rate.

2 Model Setup

Consider an insurance company whose surplus is given by a Brownian motion with drift $X_t = x + \mu t + \sigma W_t$, where W is a standard Brownian motion $\mu, \sigma > 0$. We assume that the underlying filtration \mathcal{F} is complete, right-continuous and that W is a standard \mathcal{F} -Brownian motion. Further, we model the stochastic interest rate r as a continuous time \mathcal{F} -Markov chain. For simplicity, we assume that the state space \mathcal{S} consists of only two points $\delta_1 \leq 0 < \delta_2$ and the Markov chain switches with intensities $\lambda_1 > 0$ and $\lambda_2 \geq 0$ respectively, i.e. its generator matrix is given by

$$\mathcal{G} := \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix}.$$

Further, we assume that W , r are independent but the experienced reader might note that the independence follows from the previous assumptions.

The insurance company is allowed to ask for capital injections at any time, where the accumulated capital injections until t are given by Y_t , yielding for the ex-controlled surplus X^Y :

$$X_t^Y = x + \mu t + \sigma W_t + Y_t.$$

We call a strategy Y admissible if Y is a right-continuous, non-decreasing and \mathcal{F} -adapted process which starts in zero with $Y_t \geq (-\inf\{X_s : s \in [0, t]\}) \vee 0$. We denote the class of those processes by \mathcal{A} .

As a risk measure, we consider the value of expected discounted injections, where the injected capital is discounted by the stochastic interest rate r_t . The return function corresponding to an admissible strategy $Y \in \mathcal{A}$ is given by:

$$V^Y(x, \eta) := \mathbb{E}_{x, \eta} \left[\int_0^\infty e^{-\int_0^t r_s \, ds} dY_t \right],$$

where the indices x and η indicate $X_0 = x$ and $r_0 = \eta$. We seek to minimise the total discounted injected capital, i.e. we seek to find an admissible strategy Y^* such that

$$V(x, \eta) := \inf_{Y \in \mathcal{A}} V^Y(x, \eta) = V^{Y^*}(x, \eta), \quad x \geq 0, \eta \in \mathcal{S}. \quad (1)$$

The formal corresponding Hamilton–Jacobi–Bellman equation for $i, j \in \{1, 2\}$, $i \neq j$ and $x \geq 0$ is

$$\min \left\{ \frac{\sigma^2}{2} V''(x, \delta_i) + \mu V'(x, \delta_i) - (\delta_i + \lambda_i) V(x, \delta_i) + \lambda_i V(x, \delta_j), \right. \\ \left. V'(x, \delta_i) + 1 \right\} = 0. \quad (2)$$

Notation 2.1

For the sake of convenience, we introduce the following notation

•

$$\mathcal{L}_i(f)(x) = \frac{\sigma^2}{2} f''(x) + \mu f'(x) - (\delta_i + \lambda_i) f(x)$$

for $i \in \{1, 2\}$ and a sufficiently smooth function f . \mathcal{L}_i can be applied also on $V^Y(x, \eta)$, whereas the notation $(V^Y)'(x, \eta)$ denotes the derivative with respect to x .

• We define $Y_t^0 := 0 \vee -\inf\{X_s : s \in [0, t]\}$, the corresponding return function and the ex-injection process will be denoted by $V^0(x, \eta)$ and by $X^0 := X^{Y^0}$ respectively. In the following we call the strategy Y^0 the minimal-amount strategy. This strategy is the zero-barrier strategy, i.e. we only inject capital to avoid bankruptcy.

Since a negative interest rate can lead to an infinite return function, we have to find the conditions under which the minimisation problem is well-posed. That is, we want to find an admissible strategy Y such that $V^Y(x, \eta) < \infty$ for $x \geq 0, \eta \in \mathcal{S}$.

Proposition 2.2

Assume that $\delta_1 > -\frac{\lambda_1 \delta_2}{\lambda_2 + \delta_2}$. Then, the strategy Y^0 satisfies

$$V^0(x, \eta) < \infty$$

for any $x \geq 0, \eta \in \mathcal{S}$. In particular, the stochastic control problem (1) is well-posed.

Proof: Let $x \geq 0$ and $\eta \in \mathcal{S}$. Clearly, the strategy Y^0 is independent of the stochastic interest rate process r . First we calculate the average interest rate and then we relate it to the expectation. Define the occupation time of the stochastic

interest rate in the level δ_1 by $\Lambda(t) := \int_0^t \mathbb{1}_{\{r_s = \delta_1\}} ds$ for any $t \geq 0$. Then, we have $\int_0^t r_s ds = t\delta_2 + (\delta_1 - \delta_2)\Lambda(t)$ for $t \geq 0$. Hence, we get

$$\mathbb{E}_{x,\eta} \left[\exp \left(- \int_0^t r_s ds \right) \right] = \exp(-t\delta_2) \mathbb{E}_{x,\eta} \left[\exp \left(- (\delta_1 - \delta_2)\Lambda(t) \right) \right], \quad t \geq 0.$$

Let

$$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad R := \begin{pmatrix} -\lambda_1 - \delta_1 + \delta_2 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix}.$$

From [14, p. 385] one knows

$$\mathbb{E}_{x,\eta} \left[\exp \left(- (\delta_1 - \delta_2)\Lambda(t) \right) \right] = (\mathbb{1}_{\{r_0 = \delta_1\}}, \mathbb{1}_{\{r_0 = \delta_2\}}) \cdot \exp(tR) \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad t \geq 0.$$

Defining

$$\begin{aligned} a &:= -(\lambda_1 + \lambda_2 + \delta_1 + \delta_2), \\ b &:= \sqrt{(\lambda_1 + \lambda_2 + \delta_1 - \delta_2)^2 + 4\lambda_2(\delta_2 - \delta_1)}, \\ \omega_1 &:= \delta_2 + \frac{1}{2}(a + b), \quad \omega_2 := \delta_2 + \frac{1}{2}(a - b), \end{aligned}$$

we find that

$$\exp(tR) = \frac{\omega_1 e^{t\omega_2} - \omega_2 e^{t\omega_1}}{\omega_1 - \omega_2} \cdot I + \frac{e^{t\omega_1} - e^{t\omega_2}}{\omega_1 - \omega_2} \cdot R.$$

Since,

$$\begin{aligned} (\mathbb{1}_{\{r_0 = \delta_1\}}, \mathbb{1}_{\{r_0 = \delta_2\}}) \cdot I \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= 1 \quad \text{and} \\ (\mathbb{1}_{\{r_0 = \delta_1\}}, \mathbb{1}_{\{r_0 = \delta_2\}}) \cdot R \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= (\delta_2 - \delta_1) \mathbb{1}_{\{r_0 = \delta_1\}} \end{aligned}$$

we find that

$$\begin{aligned} \mathbb{E}_{x,\eta} \left[\exp \left(- \int_0^t r_s ds \right) \right] &= \frac{e^{t(\omega_1 - \delta_2)} - e^{t(\omega_2 - \delta_2)}}{\omega_1 - \omega_2} (\delta_2 - \delta_1) \mathbb{1}_{\{r_0 = \delta_1\}} \\ &\quad + \frac{\omega_1 e^{t(\omega_2 - \delta_2)} - \omega_2 e^{t(\omega_1 - \delta_2)}}{\omega_1 - \omega_2}. \end{aligned}$$

Observe that $\omega_2 - \delta_2 < \omega_1 - \delta_2 = \frac{1}{2}(a + b) =: -c < 0$ by assumption. Hence, there is a positive constant $C > 0$ depending on $\lambda_1, \lambda_2, \delta_1, \delta_2$ such that

$$\mathbb{E}_{x,\eta} \left[\exp \left(- \int_0^t r_s ds \right) \right] \leq C \exp(-tc), \quad t \geq 0.$$

Thus, we have

$$V^0(x, \eta) \leq C \mathbb{E}_{x,\eta} \left[\int_0^\infty e^{-cs} dY_s^0 \right] < \infty.$$

□

3 The Value Function and the Optimal Strategy

In this section we aim at identifying the value function and the optimal strategy. From now on, we always assume

Assumption: $\delta_1 > -\frac{\lambda_1 \delta_2}{\lambda_2 + \delta_2}$.

Then, Proposition 2.2 yields that the stochastic control problem (1) is well-posed. Also note that this implies $\delta_1 > -\lambda_1$.

3.1 Performance of the minimal-amount injection strategy

We start our investigation by analysing the performance of the minimal-amount injection strategy Y^0 , which turns out to be optimal in some cases. We calculate its performance function $V^0(x, \eta)$ in Proposition 3.1 below. There, we also specify the conditions under which Y^0 is the optimal injection strategy.

Proposition 3.1

For $\lambda_2 > 0$ define

$$\begin{aligned} a &:= \lambda_1 + \delta_1 + \lambda_2 + \delta_2 & \text{and} & & \alpha &:= \lambda_1 + \delta_1 - \lambda_2 - \delta_2, \\ D_1 &:= \frac{a - \sqrt{\alpha^2 + 4\lambda_1\lambda_2}}{2} & \text{and} & & D_2 &:= \frac{a + \sqrt{\alpha^2 + 4\lambda_1\lambda_2}}{2}, \\ A_1 &:= \frac{\mu + \sqrt{\mu^2 + 2\sigma^2 D_1}}{\sigma^2} & \text{and} & & A_2 &:= \frac{\mu + \sqrt{\mu^2 + 2\sigma^2 D_2}}{\sigma^2}, \\ E &:= \frac{\lambda_2 + \delta_2 - D_1}{\lambda_2} & \text{and} & & F &:= \frac{\lambda_2 + \delta_2 - D_2}{\lambda_2}, \\ B_2 &:= \frac{1 - F}{A_1(E - F)} & \text{and} & & C_2 &:= \frac{E - 1}{A_2(E - F)}, \\ B_1 &:= EB_2 & \text{and} & & C_1 &:= FC_2. \end{aligned}$$

Then we have

$$\begin{aligned} V^0(x, \delta_1) &= B_1 e^{-A_1 \cdot x} + C_1 e^{-A_2 \cdot x}, \\ V^0(x, \delta_2) &= B_2 e^{-A_1 \cdot x} + C_2 e^{-A_2 \cdot x} \end{aligned}$$

for any $x \geq 0$. Moreover, $V^0 = V$ if and only if $B_1 A_1^2 + C_1 A_2^2 \geq 0$. In this case Y^0 is the optimal injection strategy.

If $\lambda_2 = 0$, the calculations become much simpler. In this case, one knows immediately

$$V^0(x, \delta_2) = \frac{\sigma^2}{\mu + \sqrt{\mu^2 + 2\sigma^2(\delta_2)}} e^{-\frac{\mu + \sqrt{\mu^2 + 2\sigma^2(\delta_2)}}{\sigma^2} x}.$$

$V^0(x, \delta_1)$ can be easily obtained via solving the differential equation

$$\mathcal{L}_1(V^0)(x, \delta_1) + \lambda_1 V^0(x, \delta_2) = 0$$

with boundary conditions $(V^0)'(0, \delta_1) = -1$ and $\lim_{x \rightarrow \infty} V^0(x, \delta_1) = 0$.

Proof: Due to the assumption on δ_1 , we have $\delta_1 \delta_2 + \delta_1 \lambda_2 + \lambda_1 \delta_2 > 0$ and, hence, $D_2 > D_1 > 0$. Also, we see that $A_1, A_2 > 0$ and $E > F$.

Additionally, we have $D_j = \frac{\sigma^2}{2} A_j^2 + \mu A_j$ for $j \in \{1, 2\}$. Now, it is easy to see that for $i, j \in \{1, 2\}$ with $i \neq j$ it holds

$$\mathcal{L}_i(V^0)(x, \delta_i) + \lambda_i V^0(x, \delta_j) = 0.$$

and the right-hand side of the claimed equality is the unique solution to these systems of ODEs with derivative -1 in $x = 0$ and vanishing at infinity. Thus, we have

$$\begin{aligned} V^0(x, \delta_1) &= B_1 e^{-A_1 \cdot x} + C_1 e^{-A_2 \cdot x}, \\ V^0(x, \delta_2) &= B_2 e^{-A_1 \cdot x} + C_2 e^{-A_2 \cdot x} \end{aligned}$$

for any $x \geq 0$. Also, $V^0(\cdot, \delta_2)$ is convex and, hence, $(V^0)'(\cdot, \delta_2) \geq -1$ which yields

$$\min\{\mathcal{L}_2(V^0)(x, \delta_2) + \lambda_2 V^0(x, \delta_1), (V^0)'(x, \delta_2) + 1\} = 0$$

for any $x \geq 0$. We see that V^0 is a \mathcal{C}^2 -function and solves the HJB equation (2) iff $(V^0)'(x, \delta_2) \geq -1$ for any $x \geq 0$.

We now show that V^0 is the value function iff $(V^0)'(x, \delta_2) \geq -1$ for any $x \geq 0$. Assume first that V^0 is the value function. Then by inspecting the HJB we find that $(V^0)'(x, \delta_2) \geq -1$ for any $x \geq 0$. Now assume that $(V^0)'(x, \delta_2) \geq -1$ for any $x \geq 0$. We find that $\{x \geq 0 : (V^0)'(x, \delta_2) = -1\} \subseteq \{0\}$ because $(V^0)'(\cdot, \delta_2)$ is strictly increasing. Applying Lemma A.4 yields

$$V^0(x, \delta_2) = \min_{Y \in \mathcal{A}} \mathbb{E}_{x, \delta_2} \left[\int_0^{T_2} e^{-\delta_2 s} dY_s + e^{-\delta T_2} V^0(X_{T_2}^Y, \delta_1) \right]$$

where $T_2 \sim \text{Exp}(\lambda_2)$ is the first time the interest rate goes back to δ_1 . Similar arguments yield

$$V^0(x, \delta_1) = \min_{Y \in \mathcal{A}} \mathbb{E}_{x, \delta_1} \left[\int_0^{T_1} e^{-\delta_1 s} dY_s + e^{-\delta T_1} V^0(X_{T_1}^Y, \delta_2) \right]$$

where $T_1 \sim \text{Exp}(\lambda_1)$ is the first time the interest rate goes back to δ_2 . Using the tower property and piecing together yields that V^0 is the value function.

However, $(V^0)''(x, \delta_2)$ has at most one zero $x_0 \geq 0$ because it is the sum of two exponential functions. Above this zero we must have $(V^0)''(x, \delta_2) \geq 0$ because V^0 is decreasing. Consequently, $(V^0)''(x, \delta_2) < 0$ on $[0, x_0]$ if such a zero x_0

exists.

Now, if $B_2 A_1^2 + C_2 A_2^2 \geq 0$, then $(V^0)''(0, \delta_2) \geq 0$ and, hence, we either have $x_0 = 0$ or $(V^0)''(x, \delta_2)$ does not have any zeros. Hence, $V^0(x, \delta_2)$ is convex and, thus, we have $(V^0)'(x, \delta_2) \geq -1$.

If $(V^0)'(x, \delta_2) \geq -1$ for any $x \geq 0$, then $0 \leq (V^0)''(0, \delta_2) = B_2 A_1^2 + C_2 A_2^2$ as claimed. \square

3.2 Recursion

One might ask why it is necessary to establish a recursion if one can tackle the problem by solving the corresponding differential equation. The problem lies in the correct choice of the optimal barrier level. It turns out that the function to minimise exhibits a complex non-linear dependence on the barrier b as a variable. Even in this two states problem it is a hard challenge to find the optimal barrier in the negative state. The complexity of the problem increases significantly with the number of states. In contrast, the recursion could be generalised to an arbitrary number of states.

In this section we construct a sequence of functions $(V_n)_{n \in \mathbb{N}}$ such that $V_{2n} \rightarrow V(\cdot, \delta_2)$ and $V_{2n+1} \rightarrow V(\cdot, \delta_1)$ uniformly together with their first two derivatives. The function V_n is actually the value function of the following modified problem: The same as the original problem but we start in δ_1 if n is odd and in δ_2 if it is even, and no more capital injections need to be made after the $(n+1)$ change in the interest rate r .

Obviously, we have to invest less in the modified problems and thus we expect that $V_n \leq V$. The optimal strategy in the modified problems are proved to be of barrier type, where the barriers are adjusted at the switching times of the interest rate.

3.2.1 Initial step:

Consider at first the auxiliary problem where we seek to minimise the value of expected discounted capital injections for the preference rate $\delta_2 > 0$ up to an exponentially distributed stopping time $T_2 \sim \text{Exp}(\lambda_2)$. Because $\delta_2 > 0$, it is immediately clear that the optimal barrier is given by 0, i.e. the optimal strategy Y^0 is to inject capital just in the case the surplus becomes negative and just as much as to shift the process back to zero. Since Y^0 and T_2 are independent, we obtain

$$\mathbb{E}_x \left[\int_0^{T_2} e^{-\delta_2 t} dY_t^0 \right] = \mathbb{E}_x \left[\int_0^\infty e^{-(\delta_2 + \lambda_2)t} dY_t^0 \right].$$

Therefore, compare for instance [8], the value function is given by

$$V_0(x) := \frac{1}{A_2} e^{-A_2 x}, \quad A_2 := \frac{\mu + \sqrt{\mu^2 + 2\sigma^2(\delta_2 + \lambda_2)}}{\sigma^2},$$

i.e. $V_0(x) = \inf_{Y \in \mathcal{A}} \mathbb{E}_x[\int_0^{T_2} e^{-\delta_2 t} dY_t]$ for $x \geq 0$.

Remark 3.2

In the sequel we construct a sequence of strategies $(V_n)_{n \in \mathbb{N}}$ such that the subsequence of even numbered members converges to $V(\cdot, \delta_2)$ and the subsequence of odd numbered members converges to $V(\cdot, \delta_1)$. This result is summarised in Theorem 3.7.

Alternatively, we can start the recursion for constructing the sequence $(V_n)_{n \in \mathbb{N}}$ by defining V_0 in such a way that it is the value function of an optimal dividend problem corresponding to the negative interest rate state δ_1 instead. Then we would have

$$V_0(x) = \frac{1}{A_1} e^{-A_1 x}$$

where

$$A_1 := \frac{\mu + \sqrt{\mu^2 + 2\sigma^2(\delta_1 + \lambda_1)}}{\sigma^2}.$$

For the sake of convenience, we additionally define

$$\tilde{A}_1 := \frac{-\mu + \sqrt{\mu^2 + 2\sigma^2(\delta_1 + \lambda_1)}}{\sigma^2}.$$

3.2.2 Further steps:

Analogously to Initial step, we denote by V_n the value function of the problem with n jumps, where after the n th jump one lands in the state with $\delta_2 > 0$ and stops the consideration at the next exponential switching time. In the following, we construct the value functions $(V_n)_{n \in \mathbb{N}}$ along with the optimal barriers b_n . Proposition A.1 points out that our definitions do actually make sense and Lemma 3.6 verifies that V_n is indeed the value function of the modified problem for every $n \in \mathbb{N}$. Due to the construction of our auxiliary problems, it is clear that in the $(2n)$ th problem we start with the $\delta_2 > 0$ state, and in the $(2n + 1)$ st problem with the $\delta_1 \leq 0$ state. Theorem 3.7 states that the sequences $(V_{2n})_{n \geq 1}$ and $(V_{2n-1})_{n \geq 1}$ converge to the value function V of the original problem in a suitable way.

It is clear, that in times of positive interest rate, it is optimal to inject as late and as less as possible. That, is we know the optimal strategy: it is a barrier strategy with barrier $b_{2n} := 0$. Then, knowing the value function of the $(2n - 1)$ st problem, we can easily calculate the value function of the $(2n)$ th problem. During times of negative interest rate, it is cheaper to inject early at once, but the optimal amount is not obvious. If the optimal strategy is a constant barrier strategy, then this barrier is independent of the surplus level. In order to simplify the calculations, we can start by finding the optimal barrier for zero initial surplus. Imagine now, we have already calculated the value function of the $(2n)$ th problem. We optimise the level of the barrier $b \geq 0$ until the next switching time $T_1 \sim \text{Exp}(\lambda_1)$. The return

function V^b , corresponding to the strategy: keep the surplus over b up to T_1 and then follow the optimal strategy from $2n$, yields

$$V^b(0) = \mathbb{E}_0 \left[\int_0^{T_1} e^{-\delta_1 s} dY_s^0 + b + e^{-\delta_1 T_1} V_{2n}(b + X_{T_1}^0) \right].$$

In order to find a b minimising the above function, we have to consider just the terms depending on b :

$$g_{2n} : b \mapsto b + \mathbb{E}_0 \left[e^{-\delta_1 T_1} V_{2n}(b + X_{T_1}^{Y^0}) \right].$$

Due to Corollary 3.4 below, V_{2n} is strictly decreasing and convex, which means that g_{2n} has a unique minimum. We choose recursively a minimum b_{2n+1} for the function g_{2n} and define recursively V_{2n+1} as the unique solution to the ODE

$$\frac{\sigma^2}{2} V_{2n+1}''(x) + \mu V_{2n+1}'(x) - (\delta_1 + \lambda_1) V_{2n+1}(x) + \lambda_1 V_{2n}(x) = 0 \quad (3)$$

for $x \geq b_{2n+1}$ with $V_{2n+1}'(b_{2n+1}) = -1$, $\lim_{x \rightarrow \infty} V_{2n+1}(x) = 0$ and

$$V_{2n+1}(x) := V_{2n+1}(b_{n+1}) + (b_{2n+1} - x), \quad x \in [0, b].$$

Also, we define V_{2n+2} as the unique solution to the ODE

$$\frac{\sigma^2}{2} V_{2n+2}''(x) + \mu V_{2n+2}'(x) - (\delta_2 + \lambda_2) V_{2n+2}(x) + \lambda_2 V_{2n+1}(x) = 0 \quad (4)$$

for $x \geq 0 = b_{2n}$ with $V_{2n+2}'(0) = -1$ and $\lim_{x \rightarrow \infty} V_{2n+2}(x) = 0$.

As we will see, $(V_n)_{n \in \mathbb{N}}$ defines a sequence of convex, decreasing \mathcal{C}^2 -functions vanishing together with their derivatives at infinity.

Let

$$\mathcal{J} := \{n \in \mathbb{N} : V_n \in \mathcal{C}^2, V_n > 0, V_n'' \geq 0, V_n' < 0, \lim_{x \rightarrow \infty} V_n(x) = 0\}$$

and note that $0 \in \mathcal{J}$. Corollary 3.4 below implies that $\mathcal{J} = \mathbb{N}$. Then, for $n \in \mathbb{N}$ it holds $g'_{2n}(b) = 1 + \mathbb{E}_0 [e^{-\delta_1 T_1} V'_{2n}(b + X_{T_1}^0)]$ with a unique zero b_{2n+1} which satisfies $b_{2n+1} = 0$ if $g'_{2n}(0) \geq 0$ or

$$-1 = \mathbb{E}_0 \left[e^{-\delta_1 T_1} V'_{2n}(b_{2n+1} + X_{T_1}^0) \right].$$

Next we will show that V_{n+1} is, indeed, twice continuously differentiable for any $n \in \mathcal{J}$. Since V_{n+1} solves the ODE (4) or (3) on $[b_{n+1}, \infty)$ and since it is linear below b_{2n+1} with slope -1 it is clear that it is a \mathcal{C}^1 -function which is twice continuously differentiable on $\mathbb{R}_+ \setminus \{b_{n+1}\}$. If $b_{n+1} = 0$, then V_{n+1} is twice continuously differentiable. If $b_{n+1} > 0$, then the second left-side derivative in b_{n+1} equals zero because V_{n+1} is linear below b_{n+1} . The next lemma observes that with our choice of b_{n+1} the right-side derivative vanishes as well if $b_{n+1} > 0$.

Lemma 3.3

Let $2n \in \mathcal{J}$. If $b_{2n+1} > 0$, then $V_{2n+1}''(b_{2n+1}) = 0$. If $b_{2n+1} = 0$, then $V_{2n+1}''(b_{2n+1}) = V_{2n+1}''(0) \geq 0$. In particular, V_{2n+1} is twice continuously differentiable.

Proof: Assume first that $b_{2n+1} > 0$ and let T be an $\text{Exp}(\lambda_1 + \delta_1)$ -distributed random variable which is independent of (X, Y^0) . Then, we have $g_{2n}'(b_{2n+1}) = 0$ and, hence,

$$\begin{aligned} -1 &= \mathbb{E}_0[e^{-\delta_1 T_1} V_{2n}'(b_{2n+1} + X_{T_1}^0)] \\ &= \frac{\lambda_1}{\lambda_1 + \delta_1} \int_0^\infty (\lambda_1 + \delta_1) e^{-t(\lambda_1 + \delta_1)} \mathbb{E}_0[V_{2n}'(b_{2n+1} + X_t^0)] dt \\ &= \frac{\lambda_1}{\lambda_1 + \delta_1} \mathbb{E}_0[V_{2n}'(b_{2n+1} + X_T^0)] \\ &= \frac{\lambda_1}{\lambda_1 + \delta_1} \int_0^\infty V_{2n}'(b_{2n+1} + y) \frac{2(\lambda_1 + \delta_1)}{\sigma^2 A_1} e^{-y \tilde{A}_1} dy \\ &= \frac{2\lambda_1}{\sigma^2 A_1} \left(\int_0^\infty V_{2n}(b_{2n+1} + y) \tilde{A}_1 e^{-\tilde{A}_1 y} dy - V_{2n}(b_{2n+1}) \right) \end{aligned}$$

where we used that the density of $X_{T_1}^0$ is $\rho(y) = \frac{2(\lambda_1 + \delta_1)}{\sigma^2 A_1} e^{-\tilde{A}_1 y}$, $y \geq 0$ given in Borodin and Salminen [4, p. 252], Formula 1.2.6. Thus, we get

$$V_{2n}(b_{2n+1}) = \frac{\sigma^2 A_1}{2\lambda_1} + \tilde{A}_1 \int_{b_{2n+1}}^\infty V_{2n}(z) e^{(b_{2n+1}-z)\tilde{A}_1} dz.$$

Rewriting the ODE (3) and inserting for $V_{2n+1}(b_{2n+1})$ the value given in (5), calculated in Proposition A.1, yields

$$\begin{aligned} \frac{\sigma^2}{2} V_{2n+1}''(b_{2n+1}) &= \mu + (\lambda_1 + \delta_1) V_{2n+1}(b_{2n+1}) - \lambda_1 V_{2n}(b_{2n+1}) \\ &= 0. \end{aligned}$$

Now assume that $b_{2n+1} = 0$. Then g_{2n} attains its minimum in 0 and $g_{2n}'(b_{2n+1}) \geq 0$. Thus, we have

$$\begin{aligned} -1 &\leq \mathbb{E}_0[e^{-\delta_1 T_1} V_{2n}'(X_{T_1}^0)] \\ &= \frac{2\lambda_1}{\sigma^2 A_1} \left(\int_0^\infty V_{2n}(y) (\tilde{A}_1) e^{-y \tilde{A}_1} dy - V_{2n}(0) \right) \end{aligned}$$

which implies that

$$V_{2n}(0) \leq \frac{\sigma^2 A_1}{2\lambda_1} + \tilde{A}_1 \int_{b_{2n+1}}^\infty V_{2n}(z) e^{(b_{2n+1}-z)\tilde{A}_1} dz.$$

Hence, we get

$$V_{2n+1}''(0) \geq 0.$$

□

Finally, we find that $\mathcal{J} = \mathbb{N}$ and, hence, V_n is a convex, twice continuously differentiable, decreasing and positive valued function.

Corollary 3.4

It holds $\mathcal{J} = \mathbb{N}$.

Proof: Let $k \in \mathcal{J}$.

Case 1: k is even. Then there is an $n \in \mathbb{N}$ such that $k = 2n$. Lemma 3.3 together with Proposition A.1 yield that V_{2n+1} is a convex twice continuously differentiable function and, hence, $k + 1 = 2n + 1 \in \mathcal{J}$.

Case 2: k is odd. Then there is $n \in \mathbb{N}$ such that $k = 2n + 1$. Proposition A.1 yields that $k + 1 = 2n + 2 \in \mathcal{J}$.

Since $0 \in \mathcal{J}$ we get $\mathcal{J} = \mathbb{N}$. □

With the preceding at hand we can now prove the (pointwise) monotonicity of the sequences $(V_{2n})_{n \in \mathbb{N}}$, $(V_{2n+1})_{n \in \mathbb{N}}$ and $(b_{2n+1})_{n \in \mathbb{N}}$.

Lemma 3.5

For any $n \in \mathbb{N}$, $x \geq 0$ we have $V_{n+2}(x) \geq V_n(x)$ and we have $b_{n+2} \leq b_n$.

Proof: Let $\mathcal{J}_2 := \{n \in \mathbb{N} : b_{n+2} \leq b_n, \forall x \geq 0 : V_{n+2}(x) \geq V_{2n}(x)\}$.

We show that $0 \in \mathcal{J}_2$. Simply observe that

$$\begin{aligned} \frac{\sigma^2}{2} V_0'' + \mu V_0' - (\lambda_2 + \delta_2) V_0 &= 0, \\ \frac{\sigma^2}{2} V_2'' + \mu V_2' - (\lambda_2 + \delta_2) V_2 + \lambda_2 V_1 &= 0, \end{aligned}$$

V_1 is strictly positive, $V_2'(0) = -1 = V_0'(0)$ and $V_2(0) > V_0(0)$. Hence, [17, XVI. Comparison Theorem, p. 139] yields that $V_2(x) > V_0(x)$ for any $x \geq 0$. Consequently, $0 \in \mathcal{J}_2$.

Now let $n \in \mathcal{J}_2$.

Case 1: n is odd. Then $n + 1$ is even and, hence, we have

$$\begin{aligned} \frac{\sigma^2}{2} V_{n+1}'' + \mu V_{n+1}' - (\lambda_2 + \delta_2) V_{n+1} + \lambda_2 V_n &= 0, \\ \frac{\sigma^2}{2} V_{n+3}'' + \mu V_{n+3}' - (\lambda_2 + \delta_2) V_{n+3} + \lambda_2 V_{n+2} &= 0, \end{aligned}$$

$V_{n+2} \geq V_n$, $V_{n+3}'(0) = -1 = V_{n+1}'(0)$ and $V_{n+3}(0) > V_{n+1}(0)$. Hence, [17, XVI. Comparison Theorem, p. 139] yields that $V_{n+3}(x) > V_{n+1}(x)$ for any $x \geq 0$. Since $b_{n+3} = 0 = b_{n+1}$ we have $n + 1 \in \mathcal{J}_2$.

Case 2: n is even. Then $n + 1$ is odd. Since $n \in \mathcal{J}_2$ we get $b_{n+3} \leq b_{n+1}$. Let W_{n+1} be the solution to the ODE

$$\frac{\sigma^2}{2} W_{n+1}'' + \mu W_{n+1}' - (\lambda_1 + \delta_1) W_{n+1} + \lambda_1 V_n = 0$$

with $\lim_{x \rightarrow \infty} W_{n+1}(x) = 0$ and $W'_{n+1}(b_{n+3}) = -1$. Then $W_{n+1} \geq V_{n+1}$ on $[b_{n+3}, \infty)$. Also, the comparison principle [17, XVI. Comparison Theorem, p. 139] yields that $V_{n+3} \geq W_{n+3}$ on $[b_{n+3}, \infty)$ and, hence, $V_{n+3} \geq V_{n+1}$ on $[b_{n+3}, \infty)$. Since V_{n+3}, V_{n+1} are linear with slope -1 on $[0, b_{n+3}]$ we get that $V_{n+3} \geq V_{n+1}$ on \mathbb{R}_+ . Thus, we have $n+1 \in \mathcal{J}_2$.

Consequently, $\mathcal{J}_2 = \mathbb{N}$ which is the claim. \square

With all the properties at hand we can show that V_n is the value function of the modified control problem introduced at the beginning of the section.

Lemma 3.6

We have

$$\min \left\{ \mathcal{L}_j(V_{2n+j})(x) + \lambda_j V_{2n+j-1}(x), V'_{2n+j}(x) + 1 \right\} = 0$$

for any $n \in \mathbb{N}, x \geq 0, j \in \{1, 2\}$. In other words, we have

$$V_{2n+j}(x) = \inf_{Y \in \mathcal{A}} \mathbb{E}_x \left[\int_0^{T_j} e^{-\delta_j s} dY_s + e^{-\delta_j T_j} V_{2n+j-1}(X_{T_j}^Y) \right]$$

where T_j is an (X, Y) -independent $\text{Exp}(\lambda_j)$ -distributed random variable.

Proof: Let $n \in \mathbb{N}, x \geq 0$ and $j \in \{1, 2\}$.

If $j = 2$, then $V'_{2n+j} + 1 \geq 0$ and

$$\frac{\sigma^2}{2} V''_{2n+j}(x) + \mu V'_{2n+j}(x) - (\lambda_j + \delta_j) V_{2n+j}(x) + \lambda_j V_{2n+j-1}(x) = 0.$$

Therefore, the claim holds. Hence, we may assume that $j = 1$. Recall that V_{2n+1} solves the differential equation (3) for $x \in [b_{2n+1}, \infty)$ and fulfils $V_{2n+1}(x) = V_{2n+1}(b_{2n+1}) + b_{2n+1} - x$ for $x \in [0, b_{2n+1})$.

If $x \geq b_{2n+1}$, then we can prove the claim like described in the first case.

Now, assume by contradiction that there is $0 \leq x_0 < b_{2n+1}$ such that

$$\frac{\sigma^2}{2} V''_{2n+1}(x_0) + \mu V'_{2n+1}(x_0) - (\delta_1 + \lambda_1) V_{2n+1}(x_0) + \lambda_1 V_{2n}(x_0) < 0$$

Let \tilde{V} be the solution to the ODE

$$\frac{\sigma^2}{2} \tilde{V}''(x) + \mu \tilde{V}'(x) - (\delta_1 + \lambda_1) \tilde{V}(x) + \lambda_1 V_{2n}(x) = 0$$

for $x \in [x_0, \infty)$ with $\tilde{V}'(x_0) = -1$ and $\lim_{x \rightarrow \infty} \tilde{V}(x) = 0$, cf. Proposition A.1.

We also define $\tilde{V}(x) := \tilde{V}(x_0) + (x_0 - x)$ for $x \in [0, x_0)$. Since $x_0 < b_{2n+1}$ Corollary A.2 yields $\tilde{V}''(x_0) \leq V''_{2n+1}(b_{2n+1}) = 0$ and the latter equality holds by Lemma 3.3. \tilde{V} is the performance function of the strategy with barrier x_0 until

time T_1 and following the optimal strategy afterwards. b_{2n+1} is chosen such that the expected discounted capital injections are minimised among barrier strategies if the initial capital is zero, i.e. $\tilde{V}(0) \geq V_{2n+1}(0)$. Thus, we get $\tilde{V}(x_0) \geq \tilde{V}_{2n+1}(x_0)$ by linearity with slope -1 . Then, we have

$$\begin{aligned} 0 &> \frac{\sigma^2}{2} V_{2n+1}''(x_0) + \mu V_{2n+1}'(x_0) - (\delta_1 + \lambda_1) V_{2n+1}(x_0) + \lambda_1 V_{2n}(x_0) \\ &= -\mu - (\delta_1 + \lambda_1) V_{2n+1}(x_0) + \lambda_1 V_{2n}(x_0) \\ &\geq \mu \tilde{V}'(x_0) - (\delta_1 + \lambda_1) \tilde{V}(x_0) + \lambda_1 V_{2n}(x_0) \\ &= -\frac{\sigma^2}{2} \tilde{V}''(x_0) \geq 0, \end{aligned}$$

which is a contradiction. Consequently, we have

$$\frac{\sigma^2}{2} V_{2n+1}''(x) + \mu V_{2n+1}'(x) - (\delta_1 + \lambda_1) V_{2n+1}(x) + \lambda_1 V_{2n}(x) \geq 0$$

for any $x \in [0, b_{2n+1})$ which yields the first part of the claim. Lemma A.4 completes the proof. \square

Finally, we come to the main statement of this section. Here, we prove that the optimal strategy for the initial control problem is indeed of barrier type.

Theorem 3.7

The sequence $(V_{2n})_{n \in \mathbb{N}}$ converges together with its first two derivatives locally uniformly to $V(\cdot, \delta_2)$ and its derivatives and the sequence $(V_{2n+1})_{n \in \mathbb{N}}$ converges together with its first two derivatives locally uniformly to $V(\cdot, \delta_1)$ and its respective derivatives.

In particular, $V(\cdot, \delta_j)$ is a convex, decreasing, positive valued C^2 -function. If $b := \lim_{n \rightarrow \infty} b_{2n} > 0$, then $V''(b, \delta_1) = 0$. The optimal strategy for the initial control problem is the function

$$Y^*(t) := \sup_{s \in [0, t]} \max\{0, -\inf_{u \in [0, s]} X(u), (b - \inf_{u \in [0, s]} X(u)) \mathbb{1}_{\{r_s = \delta_1\}}\}, \quad t \geq 0.$$

Proof: Lemma 3.5 yields that both sequences are monotone increasing and, hence, have a pointwise limit in $[0, \infty]$. Let denote those limits by

$$U_2(x) := \lim_{n \rightarrow \infty} V_{2n}(x) \quad \text{and} \quad U_1(x) := \lim_{n \rightarrow \infty} V_{2n+1}(x).$$

Since $V_{2n}(x) \leq V^0(x, \delta_2)$ and $V_{2n+1}(x) \leq V^0(x, \delta_2)$ for any $n \in \mathbb{N}$, $x \geq 0$ we get $U_1(x) \leq V^0(x, \delta_1) < \infty$ and $U_2(x) \leq V^0(x, \delta_2) < \infty$ for $x \geq 0$. Observe that we have

$$\begin{aligned} |V_{2n}''(x)| &\leq \frac{2}{\sigma^2} (\mu |V_{2n}'(x)| + (\delta_2 + \lambda_2) |V_{2n}(x)| + \lambda_2 |V_{2n-1}(x)|) \\ &\leq \frac{2}{\sigma^2} (\mu + (\delta_2 + \lambda_2) |V^0(0, \delta_2)| + \lambda_2 |V^0(0, \delta_1)|). \end{aligned}$$

Proposition A.3 yields that the convergence is local uniform for the functions and their first derivatives. Let $b := \lim_{n \rightarrow \infty} b_{2n+1}$. Since V_{2n}, V_{2n+1} solve the differential equations (4) resp. (3) (which can be trivially solved for the second derivatives) we conclude that their second derivatives converge local uniform on (b, ∞) as well. Thus, U_1, U_2 are C^2 -functions on (b, ∞) and for $x \in (b, \infty)$ we have

$$\begin{aligned} \frac{\sigma^2}{2} U_1''(x) + \mu U_1'(x) - (\lambda_1 + \delta_1) U_1(x) + \lambda_1 U_2(x) &= 0 \\ \frac{\sigma^2}{2} U_2''(x) + \mu U_2'(x) - (\lambda_2 + \delta_2) U_2(x) + \lambda_2 U_1(x) &= 0. \end{aligned}$$

Since $V_{2n+1}(x)$ are linear on $[0, b]$, we have $V_{2n+1}''(x) = 0 = U_1''(x)$ for $x \in [0, b]$. In particular, V_{2n+1} converges locally uniformly on \mathbb{R}_+ together with its first two derivatives to U_1 and its first two derivatives. Thus, the same holds for the convergence of V_{2n} to U_2 .

Finally, Lemma 3.6 yields for $i, j \in \{1, 2\}$ and $i \neq j$ that

$$\begin{aligned} \min \left\{ \mathcal{L}_i(U_i)(x) + \lambda_i U_j(x), U_i'(x) + 1 \right\} \\ = \lim_{n \rightarrow \infty} \min \left\{ \mathcal{L}_i(V_{2n+i})(x) + \lambda_i V_{2n+i-1}(x), V_{2n+i}'(x) + 1 \right\} = 0. \end{aligned}$$

Thus, (U_1, U_2) is the classical solution to the HJB-equation. Applying Lemma A.4 as in the proof of Proposition 3.1 yields that $U_1(x) = V(x, \delta_1)$ and $U_2(x) = V(x, \delta_2)$ for any $x \geq 0$. \square

In the following examples we illustrate our findings.

Example 3.8

Consider the following parameters: $\delta_1 := -0.03$, $\delta_2 := 0.01$, $\lambda_1 := 0.04$, $\lambda_2 := 0.001$, $\mu := 0.005$, $\sigma := 0.13$. The value function and the zero-barrier strategy differ considerably as shown in figure 1. This effect is most strongest while the market is in the negative state. Instead of the value function we did in fact use the approximation after 10 steps in figure 1. The second picture shows the first three pairs (solid, dotted, dashed) of functions of our approximation of the value function. We find that the optimal barrier is near 0.33. Moreover, further approximations are excluded from the picture because the additional lines are visibly indistinguishable from the dashed line.

Example 3.9

Consider the following parameters: $\delta_1 := -0.56$, $\delta_2 := 0.1$, $\lambda_1 := 0.57$, $\lambda_2 := 0$, $\mu := 0.05$ and $\sigma := 0.45$.

We have chosen $\lambda_2 = 0$ for the sake of simplicity. Consider at first V^0 , the return function corresponding to the minimal-amount strategy, i.e. we apply Y^0 in both states. In the left picture of Figure 2 one sees that the second derivative $(V^0)''(x, \delta_1)$ is negative in some interval close to 0. In particular, it holds

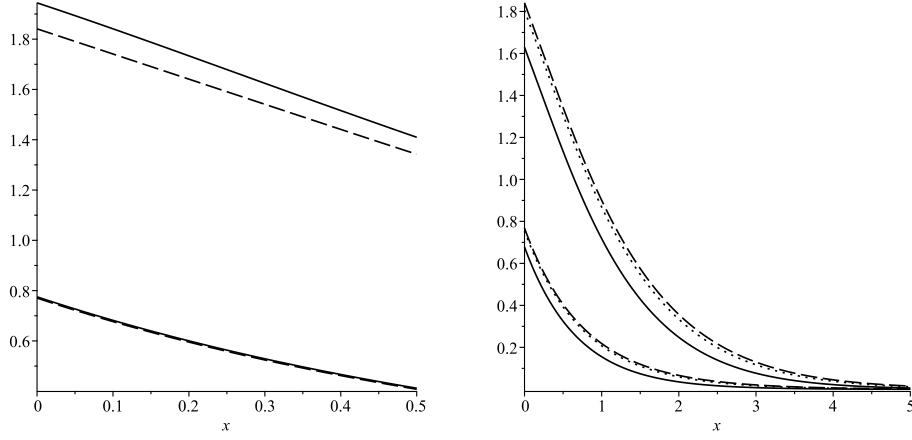


Figure 1: Left picture: Performance of the optimal strategy in δ_2 (lower dashed line) and δ_1 (upper dashed line) compared to the zero-barrier strategy (solid lines). Right picture: V_0, V_2, V_4 are in ascending the lower curves while V_1, V_3, V_5 are in ascending order the upper curves..

$(V^0)''(0, \delta_1) = -3.3077$. Thus, the strategy Y^0 cannot be optimal.

Since $\lambda_2 = 0$, we know that the value function, if starting in the state with $\delta_2 > 0$, is given by

$$V(x, \delta_2) = \frac{1}{A} e^{-Ax} \quad \text{with} \quad A = \frac{\mu + \sqrt{\mu^2 + 2\sigma^2\delta_2}}{\sigma^2}.$$

Thus, if we find the optimal barrier, we will be able to calculate the value function via the corresponding differential equation. The optimal constant barrier will minimise the expected discounted capital injections for every $x \in \mathbb{R}_+$. This means, we can choose $x = 0$. Denoting the return function corresponding to some barrier b by V^b , we obtain with $T_1 \sim \text{Exp}(\lambda_1)$:

$$\begin{aligned} V^b(0) &= b + V^b(b) = b + \mathbb{E}_0 \left[\int_0^{T_1} e^{-\delta_1 t} dY_t^0 + e^{(\delta_2 - \delta_1)T_1} V(b + X_{T_1}^0, \delta_2) \right] \\ &= b + \mathbb{E}_0 \left[\int_0^{T_1} e^{-\delta_1 t} dY_t^0 \right] + \frac{e^{-Ab}}{A} \mathbb{E} \left[e^{(\delta_2 - \delta_1)T_1} e^{-AX_{T_1}^0} \right]. \end{aligned}$$

Minimising $V^b(0)$ with respect to b , yields the condition

$$1 = e^{-Ab} \mathbb{E} \left[e^{(\delta_2 - \delta_1)T_1} e^{-AX_{T_1}^0} \right].$$

Since $\lambda_1 + \delta_1 - \delta_2 \neq 0$, we have

$$\mathbb{E} \left[e^{(\delta_2 - \delta_1)T_1} e^{-AX_{T_1}^0} \right] = \frac{\lambda_1}{\lambda_1 + \delta_1 - \delta_2} \frac{\sqrt{\mu^2 + 2\sigma^2(\lambda_1 + \delta_1)} - \sqrt{\mu^2 + 2\sigma^2\delta_2}}{\mu + \sqrt{\mu^2 + 2\sigma^2(\lambda_1 + \delta_1)}},$$

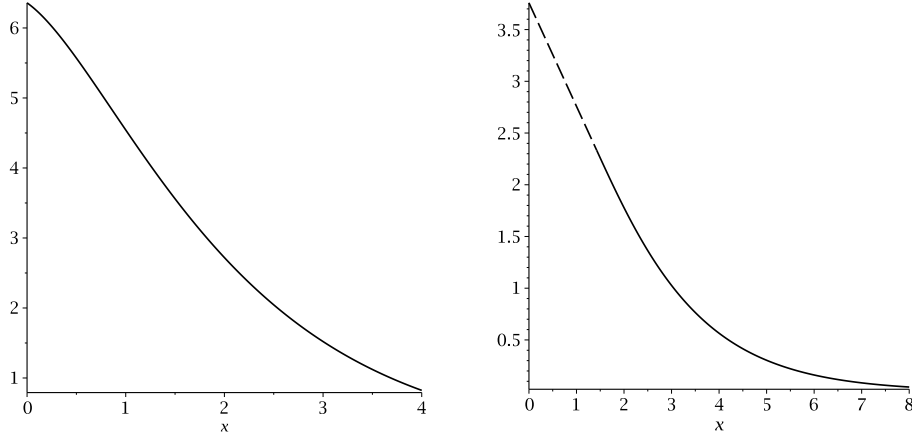


Figure 2: The non-convex structure of $V^0(x, \delta_1)$ (left picture) and the value function $V(x, \delta_1)$ (right picture).

confer for instance Borodin and Salminen, [4, p. 252], and the optimal barrier b^* is given by

$$b^* = \frac{1}{A} \ln \left(\frac{\lambda_1}{\lambda_1 + \delta_1 - \delta_2} \cdot \frac{\sqrt{\mu^2 + 2\sigma^2(\delta_1 + \lambda_1)} - \sqrt{\mu^2 + 2\sigma^2\delta_2}}{\mu + \sqrt{\mu^2 + 2\sigma^2(\delta_1 + \lambda_1)}} \right) = 1.4248 .$$

Using that $V'(b^*, \delta_1) = -1$ and $V''(b^*, \delta_1) = 0$, we can calculate the value function $V(x, \delta_1)$ by solving

$$\mathcal{L}_1(f)(x) + \lambda_1 V(x, \delta_2) = 0, \quad x \in [b^*, \infty) .$$

In the right picture of Figure 2 one sees $V(x, \delta_1)$, subdivided into the linear part on $[0, b^*]$ and the sum of two exponential functions on $(b^*, 8)$.

Remark 3.10

Increasing the number of states for the interest rate enhances realism but also the complexity of the model. Intuitively, it is clear that capital injections should be made as late as possible whenever the interest rate is positive, i.e. the barrier is equal to zero in those cases. We conjecture that for each negative state there is a barrier (which also might be zero) and that it is optimal to keep the surplus just above the barrier. In the case of only one negative state we do strongly believe that a similar approach (i.e. limiting the number of possible state switches as an approximation) could be fruitful to show the result and find a convergent algorithm as well.

A Appendix

In this section we collect auxiliary mathematical results which might be useful by themselves and are not particularly tight to the topic of the paper. First, we gather properties of a specific second order ODE, its explicit solution under the boundary conditions is given at the beginning of the proof.

Proposition A.1

Let $U : \mathbb{R}_+ \rightarrow [0, \infty)$ be a convex, decreasing and twice continuously differentiable function such that U vanishes at infinity. Let $b, \lambda > 0, \delta > -\lambda$ and V be the unique solution to the differential equation

$$\frac{\sigma^2}{2}V''(x) + \mu V'(x) - (\lambda + \delta)V(x) + \lambda U(x) = 0, \quad x \in [0, \infty)$$

with $V'(b) = -1$ and $\lim_{x \rightarrow \infty} V(x) = 0$. Then, V is strictly positive valued on $[b, \infty)$, four times continuously differentiable and

$$V(b) = \frac{1}{A} \left(1 + \frac{2\lambda}{\sigma^2} \int_b^\infty U(y) e^{\tilde{A}(b-y)} dy \right) \quad (5)$$

where

$$\begin{aligned} \psi &:= \sqrt{\mu^2 + 2\sigma^2(\delta + \lambda)} > \mu > 0, \\ A &= \frac{\mu + \psi}{\sigma^2} \quad \text{and} \quad \tilde{A} := \frac{-\mu + \psi}{\sigma^2}. \end{aligned}$$

Moreover, V' and V'' vanish at infinity. Also, the $J := \{x \in \mathbb{R}_+ : V''(x) < 0\}$ is empty or an interval containing zero and we have $V'' \geq 0 > V'$ outside of J . If $\delta \geq 0$ and $b = 0$, then $J = \emptyset$.

Proof: We have

$$\begin{aligned} V(x) &= C e^{-A(x-b)} + \frac{e^{-Ax}}{\psi} \int_b^x \lambda U(y) e^{Ay} dy + \frac{e^{\tilde{A}x}}{\psi} \int_x^\infty \lambda U(y) e^{-\tilde{A}y} dy, \\ C &:= \frac{1}{A} \left(1 + \frac{\tilde{A}}{\psi} \int_b^\infty \lambda U(y) e^{\tilde{A}(b-y)} dy \right) \end{aligned}$$

for any $x \geq 0$. Since we have $\tilde{A} > 0$, it holds $C > 0$. Observe that

$$V(b) = \frac{1}{A} \left(1 + \frac{2\lambda}{\sigma^2} \int_b^\infty U(y) e^{\tilde{A}(b-y)} dy \right)$$

as required. Consequently, $V(x) > 0$ for any $x \in [b, \infty)$. Also V is four times continuously differentiable. Clearly, V and V' vanish at infinity. Inspecting the differential equation yields that V'' vanishes at infinity.

J is an open set in \mathbb{R}_+ and, hence, countable union of disjoint open intervals. Let $I \subseteq J$ be one of those open intervals. Define $F(x) := (\lambda + \delta)V' - \lambda U'$ and by taking the derivative on the differential equation we get

$$\frac{\sigma^2}{2}V'''(x) = F(x) - \mu V''(x).$$

F is strictly decreasing on I because U is convex and $V'' < 0$ on I .

Assume by contradiction that I is non-empty and $a := \inf(I) > 0$. Then $F(a) = \frac{2}{\sigma^2}V'''(a) \leq 0$ and, hence, we have

$$\frac{\sigma^2}{2}V'''(x) = F(x) - \mu V''(x) < F(a) - \mu V''(x) \leq -\mu V''(x), \quad x \in I$$

and, hence, V'' is strictly decreasing in its zeros of \bar{I} which implies that I is unbounded and $\lim_{x \rightarrow \infty} V(x) = -\infty$. A contradiction.

Thus, either $J = \emptyset$ or $0 \in J = I$. Also, J is bounded because otherwise $V'' < 0$ everywhere and, hence, $V' \leq -1$ on $[b, \infty)$ which would imply that $\lim_{x \rightarrow \infty} V(x) = -\infty$. Thus J has the desired structure. Moreover, since $V'' \geq 0$ outside J we get V' is increasing outside J and, hence, $V' \leq 0$ outside J .

Now assume by contradiction that there is $x \geq \sup(J)$ with $V'(x) = 0$. Since V' is increasing and non-positive outside J we get $V'(y) = 0$ for any $y \geq x$ and, hence, $V''(y) = 0 = V'''(y)$ for any $y \geq x$. Thus, $F(y) = 0$ for any $y \geq x$ which implies $U'(y) = 0$ for any $y \geq x$. Thus, $U(y) = 0$ for any $y \geq x$. Hence, $V(y) = Ce^{-A(y-b)}$ for $y \geq x$ which is a contradiction to $V'(x) = 0$.

Consequently, $V'(x) < 0$ for any $x \geq \sup(J)$.

Now assume that $\delta \geq 0$, $b = 0$ and assume by contradiction that $J \neq \emptyset$. Then $F(0) = -\delta < 0$. Since F is strictly decreasing on J we get $V'''(x) = F(x) - \mu V''(x) < F(0) - \mu V''(x) \leq -\mu V''(x)$. Again, this implies that V'' is decreasing around its zeros and, hence $J = \mathbb{R}_+$. A contradiction. \square

Corollary A.2

Let $U : \mathbb{R}_+ \rightarrow [0, \infty)$ be a convex, decreasing and twice continuously differentiable function such that U vanishes at infinity. Let $\lambda > 0$, $\delta > -\lambda$ and V_b be the unique solution to the differential equation

$$\frac{\sigma^2}{2}V_b''(x) + \mu V_b'(x) - (\lambda + \delta)V_b(x) + \lambda U(x) = 0, \quad x \in [0, \infty)$$

with $V_b'(b) = -1$ and $\lim_{x \rightarrow \infty} V(x) = 0$ and denote $g(b) := V_b''(b)$ for any $b \geq 0$.

Then g is an increasing function.

Proof: We have

$$\begin{aligned}
g(b) &= -\frac{2}{\sigma^2} 2 (\mu V'_b(b) - (\lambda + \delta) V_b(b) + \lambda U(b)) \\
&= \frac{2\mu}{\sigma^2} + \frac{2\lambda}{\sigma^2} \left(\frac{\lambda + \delta}{\lambda} V_b(b) - U(b) \right) \\
&= \frac{2\mu}{\sigma^2} + \frac{2\lambda}{\sigma^2} \left(\frac{\lambda + \delta}{A\lambda} \left(1 + \frac{2\lambda}{\sigma^2} \int_b^\infty U(y) e^{\tilde{A}(b-y)} dy \right) - U(b) \right) \\
&= \frac{2\mu}{\sigma^2} + \frac{2(\lambda + \delta)}{\sigma^2 A} + \frac{2\lambda}{\sigma^2} \left(\frac{2(\lambda + \delta)}{A\sigma^2} \int_0^\infty U(z + b) e^{-z\tilde{A}} dy - U(b) \right)
\end{aligned}$$

where the third equality is implied by Proposition A.1 with A and \tilde{A} given there. Apparently, g is continuously differentiable and we have

$$\begin{aligned}
g'(b) &= \frac{2\lambda}{\sigma^2} \left(\frac{2(\lambda + \delta)}{A\sigma^2} \int_0^\infty U'(z + b) e^{-z\tilde{A}} dy - U'(b) \right) \\
&\geq \frac{2\lambda}{\sigma^2} \left(\frac{2(\lambda + \delta)}{A\sigma^2} \int_0^\infty U'(b) e^{-z\tilde{A}} dy - U'(b) \right) \\
&= \frac{2\lambda}{\sigma^2} U'(b) \left(\frac{2(\lambda + \delta)}{A\tilde{A}\sigma^2} - 1 \right) = 0
\end{aligned}$$

because $A\tilde{A} = \frac{1}{\sigma^4}(\psi^2 - \mu^2) = \frac{2(\lambda + \delta)}{\sigma^2}$. Consequently, g is an increasing function as claimed. \square

Sequences of convex \mathcal{C}^2 -functions which converge pointwise have very nice convergence behaviour. This observation is our key ingredient for our main result Theorem 3.7.

Proposition A.3

Let $(U_n)_{n \in \mathbb{N}}$ be a sequence of convex \mathcal{C}^2 -functions from \mathbb{R}_+ to \mathbb{R} which converges pointwise to some function $U : \mathbb{R}_+ \rightarrow \mathbb{R}$ and such that there is $K > 0$ with $U_n''(x) \leq K$ for any $x \geq 0$, $n \in \mathbb{N}$ and assume that $(U'_n(0))_{n \in \mathbb{N}}$ converges to some $u \in \mathbb{R}$.

Then U is a convex \mathcal{C}^1 -function and U' is Lipschitz-continuous with Lipschitz-constant at most K . Additionally, U_n, U'_n converge locally uniformly to U resp. U' .

Proof: Let $t \in [0, 1]$ and $x, y \geq 0$. Then we have

$$\begin{aligned}
U(tx + (1 - t)y) &= \lim_{n \rightarrow \infty} U_n(tx + (1 - t)y) \\
&\leq \lim_{n \rightarrow \infty} tU_n(x) + (1 - t)U_n(y) \\
&= tU(x) + (1 - t)U(y).
\end{aligned}$$

Thus, U is convex. In particular, U admits a right-derivative on $(0, \infty)$ denoted by U' .

Also, we have

$$\begin{aligned}
|U'(x) - U'(y)| &= \lim_{h \searrow 0} \left| \frac{1}{h} U(x+h) - U(x) - U(y+h) + U(y) \right| \\
&= \lim_{h \searrow 0} \lim_{n \rightarrow \infty} \left| \frac{1}{h} U_n(x+h) - U_n(x) - U_n(y+h) + U_n(y) \right| \\
&= \lim_{h \searrow 0} \lim_{n \rightarrow \infty} \left| \frac{1}{h} \int_0^h U'_n(x+z) - U'_n(y+z) dz \right| \\
&\leq K|x-y|.
\end{aligned}$$

Thus, U' is Lipschitz-continuous with constant K on $(0, \infty)$. Consequently, U is \mathcal{C}^1 on $(0, \infty)$ with derivative U' . For $h \in (0, x)$ we have

$$U'_n(x) \leq \frac{1}{h} \int_x^{x+h} U'_n(y) dy = \frac{U_n(x+h) - U_n(x)}{h} \xrightarrow{n \rightarrow \infty} \frac{U(x+h) - U(x)}{h}$$

and since this is true for any h we get $\limsup_{n \rightarrow \infty} U'_n(x) \leq U'(x)$. Also, we have

$$U'_n(x) \geq \frac{1}{h} \int_{x-h}^x U'_n(y) dy = \frac{U_n(x) - U_n(x-h)}{h} \xrightarrow{n \rightarrow \infty} \frac{U(x) - U(x-h)}{h}$$

and, hence, $\liminf_{n \rightarrow \infty} U'_n(x) \geq U'(x)$. Consequently, $U'_n(x) \rightarrow U'(x)$ for any $x > 0$.

We have

$$|U'_n(x)| \leq Kx + |U'_n(0)| \leq Kx + \sup_{n \in \mathbb{N}} U'_n(0)$$

and, hence, the dominated convergence theorem yields that $U_n \rightarrow U$ locally uniformly.

Now, let $\tilde{K} > 0$ and $\epsilon > 0$. We will show that there is $N \in \mathbb{N}$ such that that $\sup_{n \geq N} \sup_{x \in [0, \tilde{K}]} |U'_n(x) - U'(x)| \leq \epsilon$ which yields that $U'_n \rightarrow U'$ locally uniformly.

For that choose $N \in \mathbb{N}$ such that

$$\sup_{n \geq N} \sup_{x \in [0, 2\tilde{K}]} |U_n(x) - U(x)| \leq 1 \wedge \left(\frac{\epsilon}{2\tilde{K} + 2} \right)^2 =: \delta.$$

Then we have for $n \geq N$ and $x \in [0, \tilde{K}]$ with $h := \sqrt{\delta}$

$$\begin{aligned}
|U'_n(x) - U'(x)| &\leq \frac{1}{h} \int_x^{x+h} U'_n(y) dy - U'_n(x) + \left| \frac{1}{h} \int_x^{x+h} U'_n(y) dy - U'(x) \right| \\
&\leq \tilde{K}h + \left| \frac{U_n(x+h) - U_n(x)}{h} - U'(x) \right| \\
&\leq \tilde{K}h + 2\delta/h + \left| \frac{U(x+h) - U(x)}{h} - U'(x) \right| \\
&\leq 2\tilde{K}h + 2\delta/h \\
&= \epsilon \wedge (2\tilde{K} + 2) \leq \epsilon.
\end{aligned}$$

Since the estimate is independent of n and x we get the required convergence. \square

Classical verification results are straightforward and needed several times in this paper.

Lemma A.4

Let $\lambda \geq 0$, $\delta > -\lambda$, $T \sim \text{Exp}(\lambda)$ be independent of the Brownian motion W with $T \sim T|\mathcal{F}_t, T \geq t$ and $U : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ measurable. Let $V : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a twice continuously differentiable function such that

$$\min \left\{ \frac{\sigma^2}{2} V''(x) + \mu V'(x) - (\delta + \lambda) V(x) + \lambda U(x), V'(x) + 1 \right\} = 0, \quad x \geq 0$$

and such that V, V' vanish at infinity. We also assume that there is $b \geq 0$ such that $\{x \in \mathbb{R} : V'(x) = -1\} = [0, b]$ or $\{x \in \mathbb{R} : V'(x) = -1\} = \emptyset$ (in the later case we use the convention $b = 0$).

Then we have

$$V(x) = \min_{Y \in \mathcal{A}} \mathbb{E}_x \left[\int_0^T e^{-\delta s} dY_s + e^{-\delta T} U(X_T^Y) \right]$$

for any $x \geq 0$.

Proof: Let $Y \in \mathcal{A}$. Then, for some martingale M (it is a true martingale due to the boundedness of V') we have

$$\begin{aligned} e^{-(\lambda+\delta)t} V(X_t^Y) &= V(x) + \int_0^t V'(X_s^Y) e^{-(\lambda+\delta)s} dY_s + M_t \\ &\quad + \int_0^t \left(\mu V'(X_s^Y) + \frac{\sigma^2}{2} V''(X_s^Y) - (\lambda + \delta) V(X_s^Y) \right) e^{-(\lambda+\delta)s} ds \\ &\geq V(x) - \int_0^t e^{-(\lambda+\delta)s} dY_s + M_t - \int_0^t \lambda U(X_s^Y) e^{-(\lambda+\delta)s} ds. \end{aligned}$$

Solving for $V(x)$, taking the expectation and sending t to infinity yields

$$\begin{aligned} V(x) &\leq \mathbb{E}_x \left[\int_0^\infty e^{-(\lambda+\delta)s} dY_s + \int_0^\infty \lambda U(X_s^Y) e^{-(\lambda+\delta)s} ds \right] \\ &= \mathbb{E}_x \left[e^{-\delta T} U(X_T^Y) + \int_0^T e^{-\delta s} dY_s \right] \end{aligned}$$

and, hence, we have

$$V(x) \leq \inf_{Y \in \mathcal{A}} \mathbb{E}_x \left[\int_0^T e^{-\delta s} dY_s + e^{-\delta T} U(X_T^Y) \right].$$

Now, define

$$Y_t := \max\{0, b - \inf_{s \in [0, t]} X_s\}.$$

Then $Y \in \mathcal{A}$. The process Y increases only on the set $N := \{t \geq 0 : X_t = \inf_{s \in [0, t]} X_s\}$ which is known to have Lebesgue-measure zero P -a.s. Outside N we find that $X_t^Y > b$ and on the set N we find $X_t^Y \leq b$. Hence, we have $V'(X_t) = -1$ for $t \in N$ and

$$\mu V'(X_t^Y) + \frac{\sigma^2}{2} V''(X_t^Y) - (\lambda + \delta) V(X_t^Y) = -\lambda U(X_t^Y)$$

for $t \in \mathbb{R}_+ \setminus N$. Thus, for some martingale M we have

$$e^{-(\lambda+\delta)t} V(X_t^Y) = V(x) - \int_0^t e^{-(\lambda+\delta)s} dY_s + M_t - \int_0^t \lambda U(X_s^Y) e^{-(\lambda+\delta)s} ds.$$

As above, it follows

$$V(x) = \mathbb{E}_x \left[e^{-\delta T} U(X_T^Y) + \int_0^T e^{-\delta s} dY_s \right].$$

Consequently, we have

$$V(x) = \min_{Y \in \mathcal{A}} \mathbb{E}_x \left[\int_0^T e^{-\delta s} dY_s + e^{-\delta T} U(X_T^Y) \right]$$

for any $x \geq 0$. □

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